

Stochastic Description of a Bose–Einstein Condensate

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Abstract. In this work we give a positive answer to the following question: does Stochastic Mechanics uniquely define a three-dimensional stochastic process which describes the motion of a particle in a Bose–Einstein condensate? To this extent we study a system of N trapped bosons with pair interaction at zero temperature under the Gross–Pitaevskii scaling, which allows to give a theoretical proof of Bose–Einstein condensation for interacting trapped gases in the limit of N going to infinity. We show that under the assumption of strictly positivity and continuous differentiability of the many-body ground state wave function it is possible to rigorously define a one-particle stochastic process, unique in law, which describes the motion of a single particle in the gas and we show that, in the scaling limit, the one-particle process continuously remains outside a time dependent random “interaction-set” with probability one. Moreover, we prove that its stopped version converges, in a relative entropy sense, toward a Markov diffusion whose drift is uniquely determined by the order parameter, that is the wave function of the condensate.

1. Introduction

It is well-known that the wave function ψ of a spinless quantum particle defines a Nelson’s diffusion in great generality: denoting by m the mass of the particle and by V a scalar potential, so that ψ is solution of the Schrödinger equation

$$i\hbar\partial_t\psi = \left(-\frac{\hbar^2}{2m}\Delta + V\right)\psi \quad (1.1)$$

then, if V is of Rellich class and the initial kinetic energy is finite [4], there exists a weak solution X to the Stochastic Differential Equation

$$dX_t = \frac{\hbar}{m} \left(\operatorname{Re} \frac{\nabla\psi}{\psi} + \operatorname{Im} \frac{\nabla\psi}{\psi} \right) (X_t, t) dt + \left(\frac{\hbar}{m} \right)^{\frac{1}{2}} dW_t \quad (1.2)$$

where dW_t denotes the increment of a standard Brownian Motion. Notably, the diffusion X satisfies the stochastic version of the second Newton's law

$$a_N(X_t, t) = -\frac{1}{m} \nabla V(X_t, t) \quad (1.3)$$

where a_N denotes the natural mean stochastic acceleration as introduced by Nelson [13]. In addition, up to regularity assumptions, X is critical for the mean classical action functional [6] (see also [5] for a recent review and a list of interesting open problems).

If not otherwise specified, in the following capital letters will denote stochastic processes, while $\hat{X} =: (X_1, \dots, X_N)$ will stand for arrays in \mathbb{R}^{3N} and bold letters for vectors in \mathbb{R}^3 .

Now consider a system of N pair interacting copies of such a particle, with Hamiltonian

$$H_N = \sum_{i=1}^N \left(-\frac{\hbar^2}{2m} \Delta_i + V(\mathbf{r}_i) \right) + \sum_{1 \leq i < j \leq N} v(\mathbf{r}_i - \mathbf{r}_j)$$

and let Ψ_N be its ground state, that we assume strictly positive and of class C^1 (for regularity conditions on the potentials V and v implying the strictly positivity see [14], Thm.XIII.46 and Thm. XIII.47 and for those implying differentiability properties again [14], Section (XIII.11)).

We denote by \hat{X} the corresponding $3N$ -dimensional Nelson's diffusion, whose generator is also related to H_N by a Doob's transformation (see for example [15] Ch.VIII, Prop.3.9).

It satisfies the SDE

$$d\hat{X}_t = \frac{\hbar}{m} \frac{\nabla^{(N)} \Psi_N}{\Psi_N}(\hat{X}_t) dt + \left(\frac{\hbar}{m} \right)^{\frac{1}{2}} d\hat{W}_t \quad (1.4)$$

where $\nabla^{(N)}$ denotes the $3N$ -dimensional gradient and \hat{W} is a $3N$ -dimensional standard Brownian Motion. The process \hat{X} can be seen as a family of N one-particle interacting diffusions (X_1, \dots, X_N) .

If Bose–Einstein condensation occurs, the condensate is usually described by the order parameter $\phi_{GP} \in L^2(\mathbb{R}^3)$, also called wave function of the condensate, which is the minimizer of the Gross–Pitaevskii functional

$$E^{GP}[\phi] = \int \left(\frac{\hbar^2}{2m} \|\nabla \phi(r)\|^2 + V(r) |\phi(r)|^2 + \frac{g}{2} |\phi(r)|^4 \right) d\mathbf{r}$$

under the L^2 -normalization condition

$$\int_{\mathbb{R}^3} |\phi^{GP}|^2 d\mathbf{r} = 1$$

and where $g > 0$ is a parameter depending on the interaction potential v (see also next assumption h2)). Therefore ϕ_{GP} solves the stationary cubic non-linear equation (in this context called Gross–Pitaevskii equation)

$$-\frac{\hbar^2}{2m}\Delta\phi + V\phi + g|\phi|^2\phi = \lambda\phi \quad (1.5)$$

λ denoting the chemical potential.

Now we ask the following questions:

- 1) Does Stochastic Mechanics of the N -body problem associated to H_N uniquely determine a well defined stochastic process which describes the motion of a single particle in the condensate?
- 2) If such a stochastic process exists, how is it related to the order parameter ϕ_{GP} ?

In this work we give an answer to both questions. We restrict the problem to the case of the Gross–Pitaevskii scaling limit as introduced in [9], which allows to prove the existence of an exact Bose–Einstein condensation for the ground state of H^N [9, 10] (see also [1, 7, 8] for the derivation of the time-dependent Gross–Pitaevskii equation).

In Sect. 2, we introduce the one particle non-markovian diffusion and we study the relationship with the results given, within the canonical formalism, in [9, 10].

In Sect. 3, we define a suitable time-dependent random “interaction-set” and we show that, in the scaling limit, a generic particle continuously remains outside such a set with probability one in any finite time interval.

In Sect. 4, we show that the stopped one particle process converges, in a relative entropy sense, to a stopped Markovian diffusion with drift equal to $\frac{\nabla\phi_{GP}}{\phi_{GP}}$.

2. Rescaling the One-Particle Process

For simplicity of notations, in the following we will put $\hbar = m = 1$.

We firstly notice that the fixed time joint probability density of (X_1, \dots, X_N) is given by $|\Psi_N|^2$, which is invariant under spatial permutations. We can also see that, as expected, if some smoothness conditions are assumed for Ψ_N , the processes $\{X_i\}_{i=1, \dots, N}$ are equal in law :

Proposition 1. *Let Ψ_N be the ground state of H_N and assume it is strictly positive and of class C^1 . Then the three-dimensional processes $\{X_i\}_{i=1, \dots, N}$ are equal in law.*

Proof. By the symmetry of Ψ_N the joint probability density $\rho_N := |\Psi_N|^2$ is also symmetric. This implies that all marginals are identical and symmetric. Moreover, for all $k = 2, \dots, N$ and $t \geq 0$, the permutations of $(X_{i_1}(t), \dots, X_{i_k}(t))$ are identically distributed random elements.

Following [12] we observe that, if Ψ_N is of class C^1 , setting $\Psi_N =: \exp R_N$ and $i < j$, we have, ∇_j denoting the gradient with respect to the variable in the j th position

$$\nabla_i R_N(\mathbf{r}_1, \dots, \mathbf{r}_i, \dots, \mathbf{r}_j, \dots, \mathbf{r}_N) = \nabla_j R_N(\mathbf{r}_1, \dots, \mathbf{r}_j, \dots, \mathbf{r}_i, \dots, \mathbf{r}_N)$$

then

$$\begin{aligned} b_i^N(X_1, \dots, X_N) &= \nabla_i R_N(X_1, \dots, X_i, \dots, X_j, \dots, X_N) \\ &= \nabla_j R_N(X_1, \dots, X_j, \dots, X_i, \dots, X_N) \\ &\approx \nabla_j R_N(X_1, \dots, X_i, \dots, X_j, \dots, X_N) \\ &= b_j^N(X_1, \dots, X_N) \end{aligned}$$

where \approx denotes the equality in law.

Denoting by $(\hat{X}, \hat{W})(\Omega^N, \mathcal{F}^N, \mathbb{P}^N)(\mathcal{F}_t^N)_{t \geq 0}$ a solution to (1.4) we define, for any $i = 1, \dots, N$ the adapted process

$$\beta_i^N(t) := b_i^N(X_1(t), \dots, X_N(t))$$

Then, for any i , X_i satisfies the stochastic differential equation

$$dX_i(t) = \beta_i^N(t) dt + dW_i(t)$$

So, varying i from 1 to N , we get a family of three-dimensional non markovian diffusions on $(\Omega^N, \mathcal{F}^N, \mathbb{P}^N)$ with diffusion coefficient equal to 1 and identically distributed drifts. \square

We then assume, following [9],

- h1) V is locally bounded, positive and going to infinity when $|\mathbf{r}_i|$ goes to infinity. The interaction potential v is smooth, compactly supported, non negative, spherically symmetric, with finite scattering length a .
- h2) N goes to infinity and the interaction potential v satisfies the Gross-Pitaevskii scaling [9], that is

$$v(r) = v_1\left(\frac{r}{a}\right)/a^2$$

$$a = \frac{g}{8\pi N}$$

where v_1 has scattering length equal to 1. Moreover g is positive as a consequence of our assumptions on v (see h1)).

Then it is proved in [9] and [10] that $|\phi_{GP}|^2$ is in fact the L^1 limit of the one particle marginal of $|\Psi_N|^2$ and that

- A) There exists $s \in (0, 1]$, depending on the interaction potential v through the solution of the zero-energy scattering equation, such that

$$\lim_{N \uparrow \infty} \int_{\mathbb{R}^{3N}} \|\nabla_1 \frac{\Psi_N}{\phi_{GP}}\|^2 |\phi_{GP}|^2 d\mathbf{r}_1 \dots d\mathbf{r}_N = gs \int_{\mathbb{R}^3} |\phi_{GP}|^4 d\mathbf{r} \quad (2.1)$$

([10] Thm.2 Eq. (5a))

- B) Defining

$$F^N(\mathbf{r}_2, \dots, \mathbf{r}_N) := \left(\bigcup_{i=2}^N B^N(\mathbf{r}_i) \right)^c \quad (2.2)$$

where $B^N(\mathbf{r})$ denotes the open ball centered in \mathbf{r} with radius $N^{-\frac{7}{17}}$,

$$\lim_{N \uparrow \infty} \int_{\mathbb{R}^{3(N-1)}} d\mathbf{r}_2 \dots d\mathbf{r}_N \int_{F^N(\mathbf{r}_2, \dots, \mathbf{r}_N)} \|\nabla_1 \frac{\Psi_N}{\phi_{GP}}\|^2 |\phi_{GP}|^2 d\mathbf{r}_1 = 0 \quad (2.3)$$

([10], Lemma 1.) The choice of the radius of the ball B_N can be relaxed to any $N^{-\frac{1}{3}-\delta}$ with $0 < \delta < \frac{2}{9}$ (See Lemma 7.3 and its proof in [11]).

We now introduce a process X^{GP} with invariant density $\rho_{GP} := |\phi_{GP}|^2$ and try to compare it with the generic interacting non markovian diffusion X_1 .

We assume that X^{GP} is a solution of the SDE

$$dX_t^{GP} := u_{GP}(X_t^{GP}) dt + \left(\frac{\hbar}{m}\right)^{\frac{1}{2}} dW_t$$

where,

$$u_{GP} := \frac{\nabla \phi_{GP}}{\phi_{GP}}$$

Then, since ϕ_{GP} is a solution to the stationary Gross–Pitaevkii equation (1.5), a standard calculation in Stochastic Mechanics shows that Nelson acceleration of X^{GP} reads

$$a_N(X_t^{GP}) = -\frac{1}{m} \nabla \left\{ V(X_t^{GP}) + g|\phi_{GP}(X_t^{GP})|^2 \right\} \quad (2.4)$$

One could observe that now, by the non-linearity of (1.5), Doob's transformation is not expected to play any role.

It turns out that A) and B) give some important pieces of information on the relationship between the drift of the generic interacting diffusion and u_{GP} . Indeed we can observe that

$$|\phi_{GP}|^2 \|\nabla \frac{\Psi_N}{\phi_{GP}}\|^2 = |\Psi_N|^2 \left\| \frac{\nabla \Psi_N}{\Psi_N} - \frac{\nabla \phi_{GP}}{\phi_{GP}} \right\|^2$$

so that the distance between the two drifts b_1^N and u_{GP} in $L^2(|\Psi_N|^2 d\mathbf{r}_1, \dots, d\mathbf{r}_N)$ is given by the following equality

$$\begin{aligned} & \int_{\mathbb{R}^{3N}} \|b_1^N - u_{GP}\|^2 |\Psi_N|^2 d\mathbf{r}_1, \dots, d\mathbf{r}_N \\ &= \int_{\mathbb{R}^{3N}} \|\nabla_1 \frac{\Psi_N}{\phi_{GP}}\|^2 |\phi_{GP}|^2 d\mathbf{r}_1, \dots, d\mathbf{r}_N \end{aligned} \quad (2.5)$$

3. Random Interaction-Set and Stopped Processes

To explore the stochastic behavior in the scaling limit, we introduce the following time dependent random subset of \mathbb{R}^3

$$D_N(t) := \bigcup_{i=2}^N B^N(X_i(t)) \quad (3.1)$$

where $B^N(\mathbf{r})$ is again the ball with radius $N^{-\frac{7}{17}}$ centered in \mathbf{r} , and the stopping time

$$\tau^N := \inf\{t \geq 0 : X_1(t) \in D_N(t)\} \quad (3.2)$$

We recall that X_1 strongly depends on N and we do not keep explicit this dependence for simplicity of notation. Roughly, we explore the possibility that, for great N , the one particle process X_1 continuously “lives” outside the interaction-set $D_N(t)$ the most part of the time, and that its stopped version converges in some sense to the stopped version of X^{GP} .

Notice that this conjecture is not obvious. In fact, even in dimension $d = 3$, where the Lebesgue measure of $D_N(t)$ goes to zero for all t , it could happen that, asymptotically, such a set takes the form of a very complicated surface, dividing the physical three-dimensional space into smaller and smaller non connected regions. On the other side we are dealing with a random system, so that it could happen that the probability of such an event is equal to zero.

In the following proposition we prove that, in the scaling limit, a generic particle remains outside the “interaction-set”, for any finite time interval, with probability one.

Proposition 2. *Let h1) and h2) hold and the ground state Ψ_N be strictly positive and of class C^1 . Then in dimension $d = 3$, for all $t > 0$, we have*

$$\lim_{N \rightarrow \infty} \mathbb{P}(\tau^N > t | X_1(0) \notin D_N(0)) = 1 \quad (3.3)$$

and τ^N has an exponential distribution.

Proof. For all $t \geq 0$ we have, by symmetry,

$$\begin{aligned} \mathbb{P}(X_1(t) \in D_N(t)) &= \int_{\mathbb{R}^{3(N-1)}} d\mathbf{r}_2, \dots, d\mathbf{r}_N \int_{\bigcup_{i=2}^N B^N(\mathbf{r}_i)} |\Psi_N|^2 d\mathbf{r}_1 \\ &\leq (N-1) \int_{\mathbb{R}^3 \times \mathbb{R}^3} \rho_2^N(\mathbf{r}_1, \mathbf{r}_2) I_{B^N(\mathbf{r}_2)}(\mathbf{r}_1) d\mathbf{r}_1 d\mathbf{r}_2 \end{aligned}$$

where ρ_2^N denotes the two particle marginal density and $I_{B^N(\mathbf{r}_2)}(\mathbf{r}_1)$ stands for the characteristic function of the set $B_N(\mathbf{r}_2)$ in \mathbf{r}_1 .

We will show that this implies

$$\lim_{N \uparrow \infty} \mathbb{P}(X_1(t) \notin D^N(t)) = 1 \quad (3.4)$$

To see this we exploit the convergence in the trace norm of the two particle reduced density matrix (See Thm. 1 and subsequent observations in [10]).

Thus we have

$$\begin{aligned}
 & \lim_{N \uparrow \infty} \left| \int_{\mathbb{R}^3 \times \mathbb{R}^3} I_{B^N}(\mathbf{r}_2)(\mathbf{r}_1)(\rho_2^N(\mathbf{r}_1, \mathbf{r}_2) - |\phi_{GP}(\mathbf{r}_1)|^2 |\phi_{GP}(\mathbf{r}_2)|^2) d\mathbf{r}_1 d\mathbf{r}_2 \right| \\
 & \leq \lim_{N \uparrow \infty} \int_{\mathbb{R}^3 \times \mathbb{R}^3} |(\rho_2^N(\mathbf{r}_1, \mathbf{r}_2) - |\phi_{GP}(\mathbf{r}_1)|^2 |\phi_{GP}(\mathbf{r}_2)|^2) d\mathbf{r}_1 d\mathbf{r}_2| \\
 & = \lim_{N \uparrow \infty} Tr |(\gamma_2^N - |\phi_{GP}\rangle\langle\phi_{GP}|^{\otimes 2})| = 0
 \end{aligned}$$

so that

$$\begin{aligned}
 & \lim_{N \uparrow \infty} \int_{\mathbb{R}^3 \times \mathbb{R}^3} I_{B^N}(\mathbf{r}_2)(\mathbf{r}_1) \rho_2^N(\mathbf{r}_1, \mathbf{r}_2) d\mathbf{r}_1 d\mathbf{r}_2 \\
 & = \lim_{N \uparrow \infty} \int_{\mathbb{R}^3 \times \mathbb{R}^3} I_{B^N}(\mathbf{r}_2)(\mathbf{r}_1) |\phi_{GP}(\mathbf{r}_1)|^2 |\phi_{GP}(\mathbf{r}_2)|^2 d\mathbf{r}_1 d\mathbf{r}_2 \\
 & = \lim_{N \uparrow \infty} \int_{\mathbb{R}^3} |\phi_{GP}(\mathbf{r}_1)|^2 \left[\int_{\mathbb{R}^3} I_{B^N}(\mathbf{r}_1)(\mathbf{r}_2) |\phi_{GP}(\mathbf{r}_2)|^2 d\mathbf{r}_2 \right] d\mathbf{r}_1 \\
 & \leq \lim_{N \uparrow \infty} \|\phi_{GP}\|_\infty^2 (N^{-\frac{7}{17}})^3
 \end{aligned}$$

Since in our assumptions $\|\phi_{GP}\|_\infty$ is finite (see [9] Appendix A, Lemma A.5) this proves (3.4).

Now we observe that, for sufficiently large N , $\mathbb{P}(X_1(t) \notin D^N(t))$ is positive for all t , so that the conditional probability $\mathbb{P}(X_1(t) \notin D^N(t) | X_1(s) \notin D^N(s))$ is defined in elementary way for all $s < t$ and it tends to 1 when N goes to infinity.

Let now ξ_t^N denotes the distance in \mathbb{R}^3 between $X_1(t)$ and $(D^N(t))^c$. The process ξ_t^N is of the form $\xi_t^N := q(\hat{X}_t)$, where q is a continuous function from \mathbb{R}^{3N} to R^+ . Thus ξ_t^N is a continuous Markov process with respect to the natural filtration and it has a time independent transition function.

Moreover its holding time in 0 has an exponential distribution.

Putting without loss of generality t equal to 1 and following the traditional route in proving that the holding time for a right continuous Markov chain has an exponential distribution, we get, for sufficiently large N ,

$$\begin{aligned}
 \mathbb{P}(\tau^N > 1 | X_0 \notin D^N(0)) &= \mathbb{P}(\cap_{r \in \mathbb{Q} \cap [0,1]} \{\xi_r^N = 0\}) \\
 &= \lim_{n \uparrow \infty} \mathbb{P}(\cap_{i=0}^n \{\xi_{\frac{i}{n}}^N = 0\}) = \lim_{n \uparrow \infty} (p(N, n))^n
 \end{aligned}$$

where $p(N, n) := \mathbb{P}(\xi_{\frac{1}{n}}^N = 0 | \xi_0^N = 0)$.

From (3.4) and since $p(N, n)$ is less or equal to 1 for all (N, n) , we get

$$\lim_{N \uparrow \infty} \lim_{n \uparrow \infty} (p(N, n))^n = \lim_{n \uparrow \infty} \lim_{N \uparrow \infty} (p(N, n))^n = 1$$

which proves the assertion. \square

4. Relative Entropy and Convergence

We now try to compute the distance in relative entropy between the three-dimensional one-particle non-markovian diffusion X_1 and X^{GP} . To this extent we introduce a $3N$ -dimensional process \hat{X}^{GP} which satisfies a stochastic differential equation with the same diffusion coefficient as \hat{X} and drift \hat{u}_{GP} , defined by

$$\hat{u}_{GP}(\mathbf{r}_1, \dots, \mathbf{r}_N) = (u_{GP}(\mathbf{r}_1), \dots, u_{GP}(\mathbf{r}_N))$$

In this section we will assume that u_{GP} is bounded, which is a sufficient condition for applying Girsanov Theorem, that is the basic tool for defining relative entropies. (Weaker hypothesis for Girsanov Theorem can be found in [15] Ch.VIII Proposition 1.15. For the regularity of ϕ_{GP} see [9], Thm. 2.1).

We consider the measurable space $(\Omega^N, \mathcal{F}^N)$ where Ω^N is $C(\mathbb{R}^+ \rightarrow \mathbb{R}^{3N})$, and \mathcal{F}^N is its Borel sigma-algebra. We denote by $\hat{Y} := (Y_1, \dots, Y_N)$ the coordinate process and by \mathcal{F}_t^N the natural filtration.

We denote by \mathbb{P}_N and \mathbb{P}_{GP} , the measures corresponding to the weak solutions of the $3N$ - dimensional stochastic differential equations

$$\hat{Y}_t - \hat{X}_0 = \int_0^t \hat{b}^N(\hat{Y}_s) ds + \hat{W}_t \quad (4.1)$$

$$\hat{Y}_t - \hat{X}_0 = \int_0^t \hat{u}_{GP}(\hat{Y}_s) ds + \hat{W}'_t \quad (4.2)$$

where \hat{X}_0 is a random variable with probability density equal to $|\Psi_N|^2$ while \hat{W}_t and \hat{W}'_t are $3N$ -dimensional \mathbb{P}_N and \mathbb{P}_{GP} standard Brownian Motions, respectively.

In this section we use the shorthand notation $\hat{b}_s^N =: \hat{b}^N(\hat{Y}_s)$ and $\hat{u}_s^N =: \hat{u}_{GP}(\hat{Y}_s)$

The following finite energy conditions hold:

$$E_{\mathbb{P}_N} \int_0^t \|\hat{b}_s^N\|^2 ds < \infty \quad (4.3)$$

$$E_{\mathbb{P}_N} \int_0^t \|\hat{u}_s^{GP}\|^2 ds < \infty, \quad (4.4)$$

which follow from the fact that Ψ_N is the minimizer of $E^N[\Psi]$ and our hypothesis on u_{GP} .

Then, by Girsanov theorem, we have, for all $t > 0$,

$$\frac{d\mathbb{P}_N}{d\mathbb{P}_{GP}}|_{\mathcal{F}_t} = \exp \left\{ - \int_0^t (\hat{b}_s^N - \hat{u}_s^{GP}) \cdot d\hat{W}_s + \frac{1}{2} \int_0^t \|\hat{b}_s^N - \hat{u}_s^{GP}\|^2 ds \right\} \quad (4.5)$$

where $\|\cdot\|$ denotes the Euclidean norm in \mathbb{R}^{3N} . The relative entropy restricted to \mathcal{F}_t reads

$$\mathcal{H}(\mathbb{P}_N, \mathbb{P}_{GP})|_{\mathcal{F}_t} =: \mathbb{E}_{\mathbb{P}_N} \left[\log \frac{d\mathbb{P}_N}{d\mathbb{P}_{GP}} \Big|_{\mathcal{F}_t} \right] = \frac{1}{2} E_{\mathbb{P}_N} \int_0^t \|\hat{b}_s^N - \hat{u}_s^{GP}\|^2 ds \quad (4.6)$$

Since under \mathbb{P}_N the $3N$ -dimensional process \hat{Y} is a solution of (4.1) with invariant probability density $|\Psi_N|^2$, we can write, recalling also (4.3) and (4.4),

$$\begin{aligned} \frac{1}{2} E_{\mathbb{P}_N} \int_0^t \|\hat{b}_s^N - \hat{u}_s^{GP}\|^2 ds &= \frac{1}{2} \int_0^t E_{\mathbb{P}_N} \|\hat{b}_s^N - \hat{u}_s^{GP}\|^2 ds \\ &= \frac{1}{2} t \int_{\mathbb{R}^{3N}} \|\hat{b}^N(\mathbf{r}_1, \dots, \mathbf{r}_N) - \hat{u}_{GP}(\mathbf{r}_1, \dots, \mathbf{r}_N)\|^2 |\Psi_N|^2 d\mathbf{r}_1 \dots d\mathbf{r}_N \end{aligned} \quad (4.7)$$

so that, the symbol $\|\cdot\|$ now denoting the euclidean norm in \mathbb{R}^3 , we get

$$\begin{aligned} \mathcal{H}(\mathbb{P}_N, \mathbb{P}_{GP})|_{\mathcal{F}_t} &= \frac{1}{2} t \int_{\mathbb{R}^{3N}} \sum_{i=1}^N \|b_i^N(\mathbf{r}_1, \dots, \mathbf{r}_N) - u_{GP}(\mathbf{r}_i)\|^2 |\Psi_N|^2 d\mathbf{r}_1 \dots d\mathbf{r}_N \\ &= \frac{1}{2} N t \int_{\mathbb{R}^{3N}} \|b_1^N(\mathbf{r}_1, \dots, \mathbf{r}_N) - u_{GP}(\mathbf{r}_1)\|^2 |\Psi_N|^2 d\mathbf{r}_1 \dots d\mathbf{r}_N \\ &= \frac{1}{2} N E_{\mathbb{P}_N} \int_0^t \|b_1^N(\hat{Y}_s) - u_{GP}(Y_1(s))\|^2 ds \end{aligned} \quad (4.8)$$

where the symmetry of \hat{b}^N and Ψ_N has been exploited.

As a consequence, we get the sum of N identical “one-particle relative entropies”, each of them being defined by the following equality

$$\begin{aligned} \bar{\mathcal{H}}(\mathbb{P}_N, \mathbb{P}_{GP})|_{\mathcal{F}_t} &=: \frac{1}{N} \mathcal{H}(\mathbb{P}_N, \mathbb{P}_{GP})|_{\mathcal{F}_t} \\ &= \frac{1}{2} E_{\mathbb{P}_N} \int_0^t \|b_1^N(\hat{Y}_s) - u^{GP}(Y_1(s))\|^2 ds \end{aligned} \quad (4.9)$$

It is not difficult to see, by (A) and (2.5), that the one-particle relative entropy does not go to zero in the scaling limit for any $t > 0$. But things go differently for the stopped one-particle process, as we can claim in the following

Proposition 3. *Let h_1) and h_2) hold. Assume also that Ψ_N is strictly positive, of class C^1 , and that u_{GP} is bounded. Then, with τ^N defined as in (3.2), we have*

$$\lim_{N \uparrow \infty} \bar{\mathcal{H}}(\mathbb{P}_N, \mathbb{P}_{GP})|_{\mathcal{F}_{t \wedge \tau^N}} = 0 \quad (4.10)$$

Proof. Recalling (4.3) and (4.4) we can write

$$\begin{aligned}
 \bar{\mathcal{H}}(\mathbb{P}_N, \mathbb{P}_{GP})|_{\mathcal{F}_{t \wedge \tau^N}} &= \frac{1}{2} E_{\mathbb{P}_N} \int_0^{t \wedge \tau^N} \|b_1^N(\hat{Y}_s) - u_{GP}(Y_1(s))\|^2 ds \\
 &\leq \frac{1}{2} \int_0^t E_{\mathbb{P}_N} \{ \|b_1^N(\hat{Y}_s) - u_{GP}(Y_1(s))\|^2 I_{\{Y_1 \notin D_s^N\}} \} ds \\
 &= \frac{1}{2} t E_{\mathbb{P}_N} \{ \|b_1^N(\hat{Y}_s) - u_{GP}(Y_1(s))\|^2 I_{\{Y_1 \notin D_s^N\}} \} \\
 &= \frac{1}{2} t \int_{\mathbb{R}^{3N}} \|b_1^N(\mathbf{r}_1, \dots, \mathbf{r}_N) - u_{GP}(\mathbf{r}_1)\|^2 I_{F^N(\mathbf{r}_2, \dots, \mathbf{r}_N)}(\mathbf{r}_1) |\Psi_N|^2 d\mathbf{r}_1, \dots, \mathbf{r}_N \\
 &= \frac{1}{2} t \int_{\mathbb{R}^{3(N-1)}} d\mathbf{r}_2 \dots d\mathbf{r}_N \int_{F^N(\mathbf{r}_2, \dots, \mathbf{r}_N)} \left\| \nabla_1 \frac{\Psi_N}{\phi_{GP}} \right\|^2 |\phi_{GP}|^2 d\mathbf{r}_1 \quad (4.11)
 \end{aligned}$$

where we exploit (2.5). Finally, recalling (2.3), we get

$$\begin{aligned}
 &\lim_{N \uparrow \infty} \bar{\mathcal{H}}(\mathbb{P}_N, \mathbb{P}_{GP})|_{\mathcal{F}_{t \wedge \tau^N}} \\
 &= \frac{1}{2} t \lim_{N \uparrow \infty} \int_{\mathbb{R}^{3(N-1)}} d\mathbf{r}_2 \dots d\mathbf{r}_N \int_{F^N(\mathbf{r}_2, \dots, \mathbf{r}_N)} \left\| \nabla_1 \frac{\Psi_N}{\phi_{GP}} \right\|^2 |\phi_{GP}|^2 d\mathbf{r}_1 = 0 \quad (4.12)
 \end{aligned}$$

□

5. Conclusions

We have studied the Stochastic Mechanics of a system of N identical interacting trapped Bosons in the Gross–Pitaevskii scaling limit.

We have proved that the one-particle motion is described by a non-markovian diffusion X_1^N which converges, in a relative entropy sense, to the Markov diffusion X^{GP} in all random intervals of the type $[t, \tau^N)$ such that $X_1(t)$ does not belong to a time dependent random “interaction set” and τ^N is the first hitting time of such a set after t . Moreover, in the scaling limit, the first hitting time is proved to be greater than any positive time t with probability one.

For any fixed time, identifying the condensate with the fraction of particles whose position does not belong to their interaction-set at that time, we can say that the Markov diffusion X^{GP} , with drift equal to $\frac{\nabla \phi_{GP}}{\phi_{GP}}$ and Nelson’s acceleration given by (2.4), is a proper scaling limit of the stochastic process describing the motion of a generic particle of the condensate.

To study the limit behavior of averaged quantities, possibly related to quantum observables, is a non trivial problem, which deserves further work.

It could be interesting to compare our results with those given in some recent works on the stochastic descriptions of systems of interacting bosons (see for example [2] and [3]).

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